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A New Approach to Baire's Theorem and Banach Steinhaus Theorem in Linear 2-Normed Spaces

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Abstract. In this paper we construct the topological structure of linear 2-normed space. This enable us to define the concept of open sets in linear 2-normed space and derive an analogue of Baire's theorem and Banach Steinhaus theorem in linear 2-normed spaces.

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1 Introduction

The concept of a linear 2-normed space was introduced as a natural 2-metric analogue of that of a normed space. In 1963, Siegfried Gähler, a German Mathematician introduced the notion of a 2-metric space, a real valued function of point-triples on a set X, whose abstract properties were suggested by the area function for a triangle determined by a triple in Euclidean space. Many Mathematician have intensively studied this concept in the last three decades and obtained new applications of these notions in some abstract settings. In this paper, we prove an analogue of Baire's theorem and Banach Steinhauss theorem in linear 2-normed spaces X by constructing a locally convex topology for X. We now state some definitions before presenting our main results.

Let X be a linear space of dimension greater than 1 over $\mathbb{R}$. Suppose $\|,\|$ is a real valued function on $X \times X$ satisfying the following conditions:

a). $\|x, y\| = 0$ if and only if x and y are linearly independent.

b). $\|x, y\| = \|y, x\|$ for all $x, y \in X$.

c). $\|\alpha x, y\| = |\alpha| \|x, y\|$ for all $\lambda \in \mathbb{R}$ and all $x, y \in X$.

d). $\|x + y, z\| \le \|x, z\| + \|y, z\|$, for all x, y and $z \in X$.

Then $\|,\|$ is called a 2-norm on X and the pair (X, $\|,\|$) is called a linear 2-normed space. Some basic properties of linear 2-normed space can be immediately obtained as follows:

- $\|x, y\| \ge 0$, for all $x, y \in X$
- $\|x, y + \alpha x\| = \|x, y\|, \forall\ x, y \in X$ and $\forall \alpha \in R$

A standard example of a linear 2-normed space is $\mathbb{R}^2$ equipped with the 2-norm:

$\|x, y\| =$ area of the parallelogram determined by the vector x and y as the adjacent sides.

In any given 2-normed space, we can define a function p_e on X by

$$p_e(x) = \|x, e\|$$

for some $e \in X$. It is easy to see that this function satisfies the following conditions:

(1) $p_e(x + y) \le p_e(x) + p_e(y)$

(2) $p_e(\alpha x) = |\alpha| p_e(x)$

Any function defined on X and satisfying the conditions (1) and (2) is called seminorm on X. Since X is of dimension ≥ 2, corresponding to each $x \ne 0$ there exist some $e \in X$ such that x and e are linearly independent and therefore $p_e(x) \ne 0$. Thus if X is a 2-normed space, the collection $\mathbb{P} = \{p_e : e \in X\}$ forms a separating family of seminorms on X.

2 Main Results

2.1 Baire's Theorem in Linear 2-normed space

In this section we investigate the structure of open sets in linear 2-normed space and using this structure we formulate an analogue of Baire's Theorem in linear 2-normed space.

Theorem[2.1.1]: *Let X be a real linear 2-normed space. Then the subset $B_e(0,1) = \{x \in X : \|x, e\| < 1\}$ of X is convex, symmetric, balanced and absorbing.*

Proof: For any $x, y \in B_e(0,1)$ and $t \in [0,1]$,

$$\|tx + (1-t)y, e\| \le \|tx, e\| + \|(1-t)y, e\|$$

$$= |t| \|x, e\| + |(1-t)| \|y, e\|$$

$$< t+(1-t)=1$$

implying that $tx+(1-t)y \in B_e(0,1)$. Hence $B_e(0,1)$ is convex. Also for any $x \in X$, $\|-x,e\|=|-1|\|x,e\|=|x,e\|$ implies that $B_e(0,1)=-B_e(0,1)$. This shows that $B_e(0,1)$ is symmetric.

For all α with $|\alpha|\leq 1$ and $x \in B_e(0,1)$,

$$\|\alpha x, e\|=|\alpha|\|x,e\|$$

$$\leq \|x,e\| < 1$$

$$\Rightarrow \alpha x \in B_e(0,1), \forall\ x \in B_e(0,1)$$

Hence $B_e(0,1)$ is balanced.

We shall now show that $B_e(0,1)$ is absorbing. Let $x \in X$. If x and e are linearly dependent then $\|x,e\|=0<1$ and so $x \in B_e(0,1)=tB_e(0,1)$ where $t=1$. On the other hand, if we take $t=2\|x,e\|>0$, then $\|\frac{1}{t}x,e\|=\frac{1}{t}\|x,e\|=\frac{1}{2}<1$. This shows that $x \in tB_e(0,1)$ for some $t>0$. Hence $B_e(0,1)$ is absorbing. □

Theorem [2.1.2]: *Let X be a linear 2-normed space and $\mathbb{P}=\{p_e : e \in X\}$ where $p_e(x)=\|x,e\|$. Associate to each $p_e \in \mathbb{P}$ and each positive integer n set $V(p_e,n)=B_e(0,\frac{1}{n})$. Let $\mathbb{B}$ be the collection of all finite intersection of the sets $V(p_e,n)$. Then $\mathbb{B}$ is a convex balanced local base for a topology $\mathbb{T}$ on X which turns X into a locally convex space such that*

1) Every $p_e \in \mathbb{P}$ is continuous.

2) A set $E \subseteq X$ is bounded if and only if every $p_e \in \mathbb{P}$ is bounded on E.

Proof: Define a family $\mathbb{T}$ of subsets of X by $A \in \mathbb{T}$ if and only if A is a (possibly empty) union of translates of members of $\mathbb{B}$. For any $x \in X$, $\|x,e\|<n_x$ implies that $x \in n_x B_e(0,1)=n_x V(p_e,1)$ and so $X=\bigcup_{n_x} n_x V(p_e,1) \in \mathbb{T}$. clearly $\varnothing \in \mathbb{T}$ and closed under arbitrary union and finite intersection.This shows that $\mathbb{T}$ is a translation invariant topology topology on X. Since $\mathbb{B}$ is the family of finite intersection of convex and balanced subset $V(p_e,n)$ of X, each member of $\mathbb{B}$ is convex and balanced, and $\mathbb{B}$ forms a local base for $\mathbb{T}$. Next we shall prove that X is a locally convex topological vector space. Let $0 \neq x \in X$. The family $\mathbb{P}$ being separating, there exist $p_e \in \mathbb{P}$ such that $p_e(x)>0.$ Note that x is not in $V(p_e,n)$ if $np_e(x)=n\|x,e\|>1$. This shows that 0 is not in the neighbourhood $x-V(p_e,n)=x-B_e(0,\frac{1}{n})=B_e(x,\frac{1}{n})$ of x and so x is not in the closure of $\{0\}$. Since T is translation invariant, every singleton set $\{x\}=x+\{0\}$ is a closed set.

We now show that addition and scalar multiplication are continuous. Let U be a neighbourhood of 0. Then as $\mathbb{B}$ is a local base, there exist $p_{e_1}, p_{e_2}, \ldots, p_{e_m}$ in $\mathbb{P}$ and some positive integers $n_1, n_2, \ldots, n_m$ such that

$$V(p_{e_1},n_1)\cap V(p_{e_2},n_2)\cap \ldots \cap V(p_{e_m},n_m) \subseteq U.$$

Put
$$V=V(p_{e_1},2n_1)\cap V(p_{e_2},2n_2)\cap \ldots \cap V(p_{e_m},2n_m)$$

For any $z=x+y \in V+V$,

$$\|z,e_i\| = \|x+y,e_i\| \le \|x,e_i\| + \|y,e_i\| < \frac{1}{2n_i} + \frac{1}{2n_i} = \frac{1}{n_i},\ \forall\, i$$

implying that $z = x + y \in V(p_{e_i}, n_i), \forall\, i$ and so $z \in U$. Therefore $V + V \subseteq U$. This shows that vector addition is continuous. Suppose that $x \in X$, α is any scalar and U and V are as above. Then $x \in sV$ for some $s > 0$. If we take $t = \dfrac{s}{1+|\alpha|s}$ and $|\beta - \alpha| < \dfrac{1}{s}$, then

$$|\beta|t = |(\beta-\alpha)+\alpha|t \le (|\beta-\alpha| + |\alpha|)\frac{s}{1+|\alpha|s}$$

$$< \left(\frac{1}{s} + |\alpha|\right)\frac{s}{1+|\alpha|s} = 1.$$

Therefore if $y \in x + tV$ and $|\beta - \alpha| < \dfrac{1}{s}$, then as V is balanced

$$\beta y - \alpha x = \beta(y-x) + (\beta-\alpha)x \in |\beta| tV + |\beta-\alpha| sV \subseteq V + V \subseteq U$$

Thus for any neighbourhood $\alpha x + U$ of αx, there exist a neighbourhood $W = x + tV$ of x such that $\beta W \subseteq \alpha x + U$ for all β with $|\beta - \alpha| < \dfrac{1}{s}$. This proves that scalar multiplication is continuous. Hence X is a locally convex topological vector space. If $U = (-\varepsilon, \varepsilon)$ is any neighbourhood of $p_e(0) = 0$ in $\mathbb{R}$ then we can find a neighbourhood $V = V(p_e, \dfrac{1}{\varepsilon})$ of 0 in X such that $p_e(V) \subseteq U$. This shows that p_e is continuous at 0. Now let U be any neighbourhood of $p_e(x)$. Then $p_e(x) - U$ is a neighbourhood of 0 and therefore there exist some neighbourhood V of 0 in X such that $p_e(x) - p_e(V) \subseteq U$. Since V is balanced and p_e is a seminorm, it follows that $p_e(x+V) \subseteq U$. Hence p_e is continuous on X.

Now suppose that E is bounded and let $p_e \in \mathbb{P}$. Then corresponding to the neighbourhood $V(p_e, 1))$ of 0, there exist some $k > 0$ such that $E \subseteq kV(p_e, 1)$. Thus for any $x \in E$, $p_e(x) < k$. It follows that every $p_e \in \mathbb{P}$ is bounded on E.

Conversely suppose that every $p_e \in \mathbb{P}$ is bounded on E and let U be a neighbourhood of 0 in X. Then as B is a local base, there exist $p_{e_1}, p_{e_2}, \ldots, p_{e_m}$ in $\mathbb{P}$ and some positive integers $n_1, n_2, \ldots, n_m$ such that

$$V(p_{e_1}, n_1) \cap V(p_{e_2}, n_2) \cap \ldots \cap V(p_{e_m}, n_m) \subseteq U.$$

By our assumption, corresponding to each p_{e_i} there exist numbers M_i such that $P_{e_i}(x) < M_i$, for all $x \in E$ and $1 \le i \le m$. For any $x \in E$, $p_{e_i}(x) < M_i < \dfrac{n}{n_i}$, if $n > M_i n_i$. Then,

$$p_{e_i}\left(\frac{1}{n}x\right) < \frac{1}{n_i} \quad \forall\ i.$$

$$\Rightarrow \quad x \in nV(p_{e_i}, n_i)\ \forall\ i.$$

$$\Rightarrow \quad x \in nU \text{ and so } E \subseteq nU.$$

Hence E is bounded. □

Definition [2.1.3]: Let A be a convex and absorbing set in a topological vector space X. The *Minkowski's functional* μ_A of A is defined by

$$\mu_A(x) = \inf\{t > 0 : t^{-1}x \in A\} \quad for \quad x \in X.$$

Theorem [2.1.4]: *Let X be a linear 2-normed space and let $\mathbb{B}$ be the collection of all finite intersection of the sets of the form $V(p_e, n) = B_e(0, \frac{1}{n})$. Then $V = \{x \in X \ : \ \mu_V(x) < 1\}$ $\forall$ $V \in \mathbb{B}$, where μ_V is the Minkowski's functional on X.*

Proof: For any $V \in \mathbb{B}$, we can take it as $V = \bigcap_{i=1}^{m} V(p_{e_i}, n_i)$............(1)

Then for any $x \in V$, $\|x, e_i\| < \frac{1}{n_i}$ for $1 \le i \le m$. Choose t such that $n_i\|x, e_i\| < t < 1$ for all i. But then $\|\frac{x}{t}, e_i\| < \frac{1}{t}\frac{t}{n_i} = \frac{1}{n_i}$, for all i implies that $\frac{x}{t} \in V$. Thus if $x \in V$ then $\frac{x}{t} \in V$, for some $t < 1$ and so $\mu_V(x) < 1$.

Conversely if $x \notin V$, then $\frac{x}{t} \in V$ would imply that $\|x, e_i\| < \frac{t}{n_i}$ for all i. Also from (1), if $x \notin V$ then $\|x, e_i\| \ge \frac{1}{n_i}$ for some i and so $t > n_i\|x, e_i\| \ge 1$. It follows that $\mu_V(x) \ge 1$. Equivalently if $\mu_V(x) < 1$ then $x \in V$.

Hence $V = \{x \in X \ : \ \mu_V(x) < 1\}$. $\square$

Many authors have described open set in a linear 2-normed space in different ways. Here by using theorem [2.1.2], we define open and closed sets in a linear 2-normed space as follows :

Definition [2.1.5]: A subset A of a linear 2-normed space X is said to be open if for any $x \in A$ then there exist $e_1, e_2, ..., e_n$ in X and $r_1, r_2, ...r_n > 0$ such that

$$x + V(p_{e_1}, r_1) \cap V(p_{e_2}, r_2) \cap ... \cap V(p_{e_n}, r_n) = B_{e_1}(x, r_1) \cap B_{e_2}(x, r_2) \cap ... \cap B_{e_n}(x, r_n)$$

$$\subseteq A$$

where $B_{e_i}(x, r_i) = \{z \in X : \|x - z, e_i\| < r_i\}$.

A subset B of a linear 2-normed space X is said to be closed if its compliment is open in X.

Theorem [2.1.6]: *Let X be a linear 2-normed space. Then the ball $B_e(0, r) = \{x : \|x, e\| < r\}$ is open in X.*

Proof: Let $x \in B_e(0,1)$. Choose $e_m = me$ and $r_m = m(1 - \|x, e\|)$ for $m = 1,2,3,...,n$. If $y \in \cap_{m=1}^{n} B_{e_m}(x, r_m)$ then $\|y - x, e_m\| < r_m, \forall \ m$ and

$$\|y, e\| \le \|y - x, e\| + \|x, e\|$$

$$= \|y - x, \frac{e_m}{m}\| + \|x, e\|$$

$$= \frac{1}{m}\|y - x, e_m\| + \|x, e\|$$

$$< \frac{1}{m} r_m + \|x, e\|$$

$$= \frac{1}{m}\left[m(1 - \|x, e\|)\right] + \|x, e\| = 1.$$

$$\Rightarrow y \in B_e(0,1)$$

Hence $B_e(0,1)$ is open in X. □

Corollary [2.1.7]: *The ball* $B_e(a,r) = \{x : \|x - a, e\| < r\}$ *is open in a linear 2-normed space* X *for all* $a, e \in X$ *and* $r > 0$.

Proof: Let $x = a + ry \in a + rB_e(0,1) = B_e(a,r)$. Since $B_e(0,1)$ is open in X, there exist $e_1, e_2, \ldots, e_n$ in X and $r_1, r_2, \ldots, r_n > 0$ such that

$$\cap_{m=1}^{n} B_{e_m}(y, r_m) \subseteq B_e(0,1)$$

$$\Rightarrow a + r \cap_{m=1}^{n} B_{e_m}(y, r_m) \subseteq a + rB_e(0,1) = B_e(a,r).$$

$$\Rightarrow \cap_{m=1}^{n} B_{e_m}(x, R) = \cap_{m=1}^{n} B_{e_m}(a + ry, R) \subseteq B_e(a,r).$$

Hence $B_e(a,r)$ is open in X. W

Example: Let $X = \mathbb{R}^2$ be a linear 2-normed space with 2-norm defined by $\|x, y\| = |x_1 y_2 - x_2 y_1|$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ and let $e = (e_1, e_2)$. Then,

$$B_e(0,1) = \{(x_1, x_2) : \|x, e\| < 1\}$$

$$= \{(x_1, x_2) : |x_1 - x_2| < 1\}$$

$$= \{(x_1, x_2) : x_1 - 1 < x_2 < x_1 + 1\} \text{ is open in } X.$$

Definition [2.1.8]: A sequence $\{x_n\} \to x$ in a linear 2-normed space X if for any open set V containing 0 there exist a positive integer N such that $x_n - x \in V$ for all $n \geq N$.

Theorem [2.1.9]: *A sequence* $\{x_n\} \to x$ *in a 2-normed space* X *if and only* $\lim_{n\to\infty} \|x_n - x, e\| = 0$, *for all* $e \in X$.

Proof: Consider the open set $V = B_e\left(0, \frac{1}{n}\right)$ containing 0 and for any $e \in X$. If the sequence $\{x_n\}$ converges to x then we can find some positive integer N such that $x_n - x \in B_e\left(0, \frac{1}{n}\right)$ for all $n \geq N$. Then

$$\|x_n - x, e\| < \frac{1}{n}, \forall\ n \geq N \text{ and } e \in X.$$

Letting $N \to \infty$, we get $\lim_{n\to\infty} \|x_n - x, e\| = 0$, for all $e \in X$.

Conversely, suppose that $\lim_{n\to\infty} \|x_n - x, e\| = 0$, for all $e \in X$ and V is any open set containing 0. Then there exist $e_1, e_2 \ldots e_n$ in X and $r_1, r_2, \ldots r_n$ such that

$$B_{e_1}(0, r_1) \cap B_{e_2}(0, r_2) \cap \ldots \cap B_{e_n}(0, r_n) \subseteq V.$$

But then by our assumption corresponding to each $r_i > 0$, there exist a positive integer N_i such that

$$\|x_n - x, e\| < r_i, \forall\, n \geq N_i \ \textit{and for all}\ e = e_i.$$

In otherwords, $x_n - x \in B_{e_i}(0, r_i), \forall\, n \geq N = \max_i(N_i)\ \textit{and}\ \forall i$. It follows that $x_n - x \in V, \forall n \geq N.$ Hence $\{x_n\} \to x.$ $\square$

Definition [2.1.10]: A sequence $\{x_n\}$ in a linear 2-normed space X is said to be Cauchy sequence if there exist two linearly independent elements y and z such that $\lim_{m,n\to\infty}\|x_n - x_m, y\| = 0$ and $\lim_{m,n\to\infty}\|x_n - x_m, z\| = 0.$

Definition [2.1.11]: Let X be a linear 2-normed space and $A \subseteq X$. Then a point $x \in X$ is called limit point of A in X if for any open set U containing x, $A \cap (U - \{x\}) \neq \emptyset.$

Theorem [2.1.12]: *Let X be a linear 2-normed space and $A \subseteq X.$ If x is a limit point of A then corresponding to each $e \in X$ there exist a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty}\|x_n - x, e\| = 0.$*

Proof: If x is a limit point of A then corresponding to each open ball $B_e\left(x, \frac{1}{n}\right)$ we can choose $x_n \in A \cap \left(B_e\left(x, \frac{1}{n}\right) - \{x\}\right)$. Then $x_n \in A$ and $\|x_n - x, e\| < \frac{1}{n}$ for all $n.$ Thus for any $e \in X,$ there exist $\{x_n\}$ in A such that $\lim_{n\to\infty}\|x_n - x, e\| = 0.$

Definition [2.1.13]: Let X be a linear 2-normed space and $A \subseteq X.$ A point $x \in X$ is called a closure point of A if every open set containing x intersects A. The set of all closure points of A, denoted by $\overline{A}$ is called closure of $A.$

Definition [2.1.14]: Let X be a linear 2-normed space and $A \subseteq X.$ Then A is said to be dense in X if every open set U in X intersects A.

Definition [2.1.15]: A linear 2-normed space in which every Cauchy sequence is convergent, is called a 2-Banach space or a complete space.

We now prove the main objective of this section :

Theorem [2.1.12][Analogue of Baire's Theorem in Linear 2-normed space]: *Let X be a 2-Banach space. Then intersection of a countable number of dense open subsets of X is dense in X.*

Proof: Let $V_1, V_2, \ldots$ be dense open subsets of $X.$ For $x_0 \in X,$ consider an arbitrary non-empty open subset B_0 of X containing $x_0.$ Then there exist $f_1, f_2, \ldots, f_i$ in X and $p_1, p_2, \ldots, p_i > 0$ such that

$$B_{f_1}(x_0, p_1) \cap B_{f_2}(x_0, p_2) \cap \ldots \cap B_{f_i}(x_0, p_i) \subseteq B_0.$$

Since V_1 is dense in X and $\cap_{k=1}^{i} B_{f_k}(x_0, p_k)$ is open, $V_1 \cap \left(\cap_{k=1}^{i} B_{f_k}(x_0, p_k)\right) \neq \emptyset.$ Choose an element $x_1 \in V_1 \cap \left(\cap_{k=1}^{i} B_{f_k}(x_0, p_k)\right)$ Then as $V_1 \cap \left(\cap_{k=1}^{i} B_{f_k}(x_0, p_k)\right)$ is open in X , we can find an open set B_1 containing x_1 such that $\overline{B_1} \subseteq V_1 \cap \left(\cap_{k=1}^{i} B_{f_k}(x_0, p_k)\right) \subseteq V_1 \cap B_0.$ B_1 being an open set containing x_1, there exist $g_1, g_2, \ldots, g_j$ in X and $q_1, q_2, \ldots, q_j > 0$ such that

$$B_{g_1}(x_1, q_1) \cap B_{g_2}(x_1, q_2) \cap \ldots \cap B_{g_j}(x_1, q_j) \subseteq B_1.$$

Note that V_2 is dense in X and $\cap_{k=1}^{j} B_{g_k}(x_1, q_k)$ is open in $X.$ Consequently, $V_2 \cap \left(\cap_{k=1}^{j} B_{g_k}(x_1, q_k)\right) \neq \emptyset.$ Let $x_2 \in V_2 \cap \left(\cap_{k=1}^{j} B_{g_k}(x_1, q_k)\right)$ Then as above, we can choose an open set B_2 containing x_2 such that

$\overline{B_2} \subseteq V_2 \cap \left(\cap_{k=1}^{j} B_{g_k}(x_1, q_k)\right) \subseteq V_2 \cap B_1$. Thus proceeding inductively we can find a sequence $\{x_n\}$ such that $x_n \in V_{m+1} \cap \left(\cap_{i=1}^{k} B_{e_i}(x_m, r_i)\right)$ for all $n > m$ and a decreasing sequence $\{R_n\}$ of positive real numbers such that $R_n < \frac{1}{n}$. where $R_1 = max\{p_1, p_2, ..., p_i\}$, $R_2 = max\{q_1, q_2, ..., q_j\}$, . . . , $R_m = max\{r_1, r_2, ..., r_k\}$.

$$||x_n - x_m, e_i|| < R_m < \frac{1}{m}, \forall\, n > m \; and \; \forall i$$

$$\Rightarrow ||x_n - x_r, e_i|| \le ||x_n - x_m, e_i|| + ||x_m - x_r, e_i||$$

$$< \frac{1}{m} + \frac{1}{m} = \frac{2}{m}, \forall\, n, r > m \; and \;\; \forall\, i.$$

If we let $m \to \infty$, we obtain $\lim_{n,r\to\infty} ||x_n - x_r, e|| = 0$, $\forall\, e \in span\{e_1, e_2, ..., e_k\}$. This shows that $\{x_n\}$ is a Cauchy sequence in a 2-Banach space X and hence there exist some $x \in X$ such that $x_n \to x$ in X. Since $x_n \in B_m, \forall\, n \ge m$, it follows that $x \in \overline{B_m}$ and as $\overline{B_m} \subseteq V_m \cap B_0$ for $m = 1,2,3...$ we see that $x \in \left(\cap_{m=1}^{\infty} V_m\right) \cap B_0$. Hence B_0 intersects $\cap_{m=1}^{\infty} V_m$ and therefore dense in X. W

2.2 Banach Steinhaus Theorem in Linear 2-normed space

In this section, we will consider linear operators defined on a linear 2-normed space into a linear 2-normed space. We will formulate Banach Steinhauss Theorem for a family of continuous linear operators.

Definition [2.2.1]: Let X and Y be linear 2-normed spaces over $\mathbb{R}$. Then a linear map $T : X \to Y$ is continuous at x if for any open ball $B_d(T(x), R)$ in Y there exist an open ball $B_e(x, r)$ in X such that $T(B_e(x, r)) \subseteq B_d(T(x), R)$. In otherwords for any $d \in Y$ and $R > 0$, there exist some $e \in X$ and $r > 0$ such that $||T(y) - T(x), d|| < R$ whenever $||y - x, e|| < r$ and $\forall\, y, x \in X$

Theorem [2.2.2]: *Let X and Y be linear 2-normed spaces over $\mathbb{R}$. If a linear operator $T : X \to Y$ is continuous at 0 then it is continuous on X.*

Proof: Assume that the linear operator $T : X \to Y$ is continuous at 0. For any open ball $B_d(0, R)$ in Y , we can find an open ball $B_e(0, r)$ such that

$$T(B_e(0, r)) \subseteq B_d(0, R)$$

Then by linearity, $T(y) - T(x) \in B_d(0, R)$ whenever $y - x \in B_e(0, r)$. Thus if $y \in x + B_e(0, r) = B_e(x, r)$ then $T(y) \in T(x) + B_d(0, R) = B_d(T(x), R)$. Hence

$$T(B_e(x, r)) \subseteq B_d(T(x), R)$$

implying that T is continuous on X.

Definition [2.2.3]: Let X and Y be linear 2-normed spaces over $\mathbb{R}$ and $T : X \to Y$ be a linear operator. The operator T is said to be sequentially continuous at $x \in X$ if for any sequence $\{x_n\}$ of X converging to x we have $T(x_n) \to T(x)$.

Theorem [2.2.4]: *Every continuous linear map T from a linear 2-normed space X into a linear 2- normed space Y is sequentially continuous on X.*

Proof: Let $T : X \to Y$ be continuous at $x \in X$. If $B_d(T(x), R)$ is any open ball in Y ,then by the continuity of T , there exist some open ball $B_e(x, r)$ in X. such that

$$T\big(B_e(x,r)\big) \subseteq B_d(T(x),R) \qquad (1)$$

Let $\{x_n\}$ be any sequence in X such that $x_n \to x$ in X. Then corresponding to the open ball $B_e(x,r)$, there exist some $K > 0$ such that

$$x_n \in B_e(x,r), \forall\, n \geq K \qquad (2)$$

$$\Rightarrow. \ \|x_n - x, e\| < r, \ \forall\, n \geq K$$

(1) and (2) shows that $T(x_n) \in B_d(T(x),R), \forall\, n \geq K$

$$\Rightarrow \|T(x_n) - T(x), d\| < R, \ \forall\, n \geq K$$

Since $B_d(T(x),R)$ is arbitrary, it follows that $T(x_n) \to T(x)$. Hence T is sequentially continuous on X.

Theorem [2.2.5]: *Let X and Y be linear 2-normed spaces over $\mathbb{R}$. If X is finite dimensional, then every linear map from X into Y is sequentially continuous.*

Proof : Let X be finite dimensional and $T : X \to Y$ be linear. If $X = \{0\}$ then there is nothing to prove. Let now $X \neq \{0\}$ and $\{e_1, e_2, ..., e_m\}$ be a basis for X. For a sequence $\{x_n\}$ in X, let $x_n = a_{n,1}e_1 + a_{n,2}e_2 + ... + a_{n,m}e_m$ where $a_{nj} \in \mathbb{R}$. If $x_n \to x = a_1e_1 + a_2e_2 + ... + a_me_m$ in X, then

$$\|x_n - x, e_j\| = \|(a_{n,1} - a_1)e_1 + (a_{n,2} - a_2)e_2 + ... + (a_{n,m} - a_m)e_m, e_j\|$$

$$= \|(a_{n,j} - a_j)e_j + y^j, e_j\|$$

$$where\ y^j \in Y_j = span\{e_i : i = 1,2,...m\ and\ i \neq j\}$$

$$= |a_{n,j} - a_j| \|e_j + y_j, e_j\|, where y_j = \frac{1}{|a_{n,j} - a_j|} y^j$$

$$\geq |a_{n,j} - a_j|\, dist(e_j, Y_j),$$

$$where\ dist(e_j, Y_j) = \inf\{\|y, e_j\| : y \in Y_j\}$$

$$\Rightarrow \ |a_{n,j} - a_j| \leq \frac{\|x_n - x, e_j\|}{dist(e_j, Y_j)} \to 0\ as\ n \to \infty\ and\ \forall\ j.$$

$$\Rightarrow \ a_{n,j} \to a_j \ \ \forall\ j.$$

By the linearity of T, it then follows that

$$T(x_n) = T(a_{n,1}e_1 + a_{n,2}e_2 + ... + a_{n,m}e_m)$$

$$= a_{n,1}T(e_1) + a_{n,2}T(e_2) + ... + a_{n,m}T(e_m)$$

$$\to a_1T(e_1) + a_2T(e_2) + ... + a_mT(e_m)$$

$$= T(a_1e_1 + a_2e_2 + ... + a_me_m)$$

$$= T(x)$$

Thus every linear map T from X to Y is sequentially continuous. □

Definition [2.2.6]: Let X and Y be two real linear 2-normed spaces, $\{T_\lambda\}_{\lambda \in \Lambda}$ a family of linear operator from X to Y. We say that $\{T_\lambda\}_{\lambda \in \Lambda}$ is equi-continuous if for any neighbourhood $B_d(0,R)$ in Y there exist some $B_e(0,r)$ in

X such that

$$T_\lambda(B_e(0,r)) \subseteq B_d(0,R), \ \forall\, \lambda \in \Lambda.$$

In otherwords if for any $d \in Y$ and $R > 0$, there exist $e \in X$ and $r > 0$ such that $\|T(x), d\| < R,$ whenever $\|x, e\| < r$ and $\forall \lambda \in \Lambda.$

Definition [2.2.7]: A subset E of a linear 2-normed space X is said to be locally bounded if there exist some $e \in X - \{0\}$ and $r > 0$ such that $E \subseteq B_e(0, r).$

A subset E of a linear 2-normed space X is bounded if for any open ball $B_e(0,r)$ there exist some $t > 0$ such that $E \subseteq tB_e(0,r) \subset B_e(0,R).$

A linear map $T : X \to Y$ is bounded if it maps bounded set into bounded set.

Theorem [2.2.8]: *Suppose X and Y are linear 2-normed spaces over $\mathbb{R}$. Let $\{T_\lambda\}_{\lambda\in\Lambda}$ be an equi-continuous collection of linear mappings from X into Y and B be a bounded subset of X. Then $T_\lambda(B)$ is a bounded subset of Y for all $\lambda \in \Lambda.$, that is, $\{T_\lambda\}_{\lambda\in\Lambda}$ is equi-bounded.*

Proof : Let $\{T_\lambda\}_{\lambda\in\Lambda}$ be an equi-continuous collection of linear mappings from X into $Y.$ For any open ball $B_d(0,R)$ in Y, we can find an open ball $B_e(0,r)$ in X such that $T_\lambda(B_e(0,r)) \subseteq B_d(0,R) \ \forall\, \lambda \in \Lambda.$

$$\Rightarrow \|T_\lambda(x), d\| < R, \textit{ whenever } \|x, e\| < r \textit{ and } \forall\, \lambda \in \Lambda \text{..........}(1)$$

Since B is bounded, corresponding to the open ball $B_e(0,r)$ there exist some $t > 0$ such that

$$B \subseteq tB_e(0,r) \text{..............}(2)$$

If $x \in B$ then $\|\frac{x}{t}, e\| < r.$ But then from (1), we obtain

$$\|T_\lambda(x), d\| < tR = R^1, \ \forall \lambda \in \Lambda, \ d \in Y \textit{ and } \ x \in B.$$

This shows that $T_\lambda(B)$ is a bounded subset of Y for all $\lambda \in \Lambda.$, that is, $\{T_\lambda\}_{\lambda\in\Lambda}$ is equi-bounded.

Theorem [2.2.9] [Banach Steinhauss Theorem in Linear 2-normed space]:

Let X and Y be linear 2-normed spaces over $\mathbb{R}$. If X is a 2-Banach space and $\{T_\lambda\}_{\lambda\in\Lambda}$ is a family of continuous linear operator from X to Y such that for any $x \in X$, there exist $c_x > 0$ such that

$$\|T_\lambda(x), y\| < c_x \|x, e\|, \forall\, \lambda \in \Lambda, \ y \in Y \textit{ and } e \notin \textit{span}\{x\} \text{..........}(1)$$

then the family $\{T_\lambda\}_{\lambda\in\Lambda}$ is equi-continuous.

Proof: Let $B_d(0,R)$ be any open ball in Y. Note that $B_d(0,R)$ is absorbing in $Y.$ choose a positive real number r such that $\overline{B_d(0,r)} + \overline{B_d(0,r)} \subseteq B_d(0,R).$

$$\textit{Define } A_n = \left\{x \in X : T_\lambda(x) \in n\overline{B_d(0,r)}, \forall\, \lambda \in \Lambda\right\}$$

$$= \left\{x \in X : \frac{x}{n} \in T_\lambda^{-1}\left(\overline{B_d(0,r)}\right), \forall\, \lambda \in \Lambda\right\}$$

$$= \left\{x \in X : x \in nT_\lambda^{-1}\left(\overline{B_d(0,r)}\right), \forall\, \lambda \in \Lambda\right\}$$

$$= \cap_{\lambda\in\Lambda} nT_\lambda^{-1}\left(\overline{B_d(0,r)}\right)$$

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Then A_n is closed for all n and by using the given condition (1), we obtain $X = \cup_{n\in N} A_n$. Since X is a 2-Banach space, Baire's theorem shows that atleast one of A_n has non-empty interior. Let x_0 be an interior point of A_{n_0}. Then there exist an open ball $B_e(0,t)$ such that

$$x + B_e(0,t) \subseteq B_e(x,t) \subseteq A_{n_0}$$

$$\Rightarrow T_\lambda\left(B_e(0,t)\right) \subseteq T_\lambda(A_{n_0}) - T_\lambda(x)$$

$$\subseteq n_0\overline{B_d(0,r)} - n_0\overline{B_d(0,r)}$$

$$= n_0\left(\overline{B_d(0,r)} + \overline{B_d(0,r)}\right)$$

$$\subseteq n_0 B_d(0,R), \ \forall \lambda \in \Lambda.$$

$$\Rightarrow \ T_\lambda\left(B_e(0,\frac{t}{n})\right) \subseteq B_d(0,R), \ \lambda \in \Lambda.$$

This shows that $\{T_\lambda\}_{\lambda\in\Lambda}$ is equi-continuous and hence equi-bounded.

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